A reduced fast component-by-component construction of (polynomial) lattice points

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1 Introduction and Motivation

2 Tractability

3 The reduced CBC construction

4 The reduced fast CBC construction

5 Concluding remarks
Introduction and Motivation
Consider integration of functions on $[0, 1]^s$, 

$$ I_s(f) = \int_{[0, 1]^s} f(x) \, dx, $$

where $f \in \mathcal{H}$, and $\mathcal{H}$ is some Banach space.

Approximate $I_s$ by a QMC rule,

$$ I_s(f) \approx Q_{N,s}(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(x_k), $$

where $\mathcal{P}_N = \{x_0, \ldots, x_{N-1}\}$. 

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A reduced fast CBC construction

ICERM IBC 2014 4
Worst case error in Banach space $\mathcal{H}$ with respect to $\mathcal{P}_N = \{x_0, \ldots, x_{N-1}\}$:

$$e_{N,s}(\mathcal{H}, \mathcal{P}_N) := \sup_{f \in \mathcal{H}, \|f\| \leq 1} |l_s(f) - Q_{N,s}(f)|.$$ 

Need $\mathcal{P}_N$ that makes $e_{N,s}(\mathcal{H}, \mathcal{P}_N)$ small.
Weighted Korobov space: $\mathcal{H}_{s,\alpha,\gamma} = \text{space of continuous functions } f \text{ such that } \|f\|_{s,\alpha,\gamma} < \infty$, where

$$
\|f\|_{s,\alpha,\gamma}^2 = \sum_{h \in \mathbb{Z}^s} \rho_{\alpha,\gamma}(h)^{-1} |\hat{f}(h)|^2,
$$

and where $\hat{f}(h) = \int_{[0,1]^s} f(t) \exp(-2\pi i h \cdot t) \, dt$ is the $h$-th Fourier coefficient of $f$.

Furthermore, $\rho_{\alpha,\gamma}(h) = \prod_{j=1}^s \rho_{\alpha,\gamma_j}(h_j)$, and

$$
\rho_{\alpha,\gamma}(h) = \begin{cases} 
1 & h = 0, \\
\gamma |h|^{-\alpha} & h \neq 0.
\end{cases}
$$

$\alpha$ is the “smoothness parameter”,

$1 = \gamma_1 \geq \gamma_2 \geq \ldots > 0$ are the coordinate weights.
Here: $\mathcal{P}_N = \{x_0, \ldots, x_{N-1}\}$ is a lattice point set with generating vector $\mathbf{z} = (z_1, \ldots, z_s) \in \{1, \ldots, N - 1\}^s$.

Points of $\mathcal{P}_N$:

$$x_n = (x_{n,1}, \ldots, x_{n,s})$$

with

$$x_{n,j} = \left\{ \frac{nz_j}{N} \right\}.$$
For the Korobov space $\mathcal{H}_{s,\alpha,\gamma}$, and for a lattice point set $\mathcal{P}_N$, we have an explicit formula for $e^2(\mathcal{H}_{s,\alpha,\gamma},\mathcal{P}_N)$.

$$e^2_N(\mathcal{H}_{s,\alpha,\gamma},\mathcal{P}_N) = e^2_{N,s,\alpha,\gamma}(\mathbf{z}) := \sum_{h \in \mathcal{D}(\mathbf{z}) \setminus \{0\}} \rho_{\alpha,\gamma}(h),$$

where

$$\mathcal{D}(\mathbf{z}) := \{ \mathbf{h} \in \mathbb{Z}^s : \mathbf{h} \cdot \mathbf{z} \equiv 0 \ (N) \}.$$
Finite formula:

\[
e^{2}_{N,s,\alpha,\gamma}(z) = -1 + \frac{1}{N} \sum_{n=0}^{N-1} \prod_{j=1}^{s} \left( 1 + \gamma_j \varphi_{\alpha} \left( \left\{ \frac{n z_j}{N} \right\} \right) \right),
\]

where \( \varphi_{\alpha} \left( \frac{k}{N} \right) \) can be precomputed for all values of \( k = 0, \ldots, N - 1 \).

If \( \alpha = 2k, \ k \in \mathbb{N} \), \( \varphi_{\alpha} \) is a constant multiple of the Bernoulli polynomial of degree \( \alpha \).
• All that remains is to find “good” \( z \in \{1, \ldots, N - 1\}^s \).
• Rather big search space! (e.g., \( N = 10000 \) and \( s = 20 \)).
• Component by component (CBC) construction: construct \( z_j \) one at a time.
  Size of search space is \( N - 1 \) per component.
• Can do fast CBC (Cools & Nuyens), computation cost of \( \mathcal{O}(sN \log N) \).
• Computation cost of \( \mathcal{O}(sN \log N) \) can still be demanding for big \( N, s \).
• Might want to have big \( N, s \) simultaneously.
Tractability
Let $e(N, s)$ be the $N$th minimal (QMC) worst-case error,

$$e(N, s) = \inf_{\mathcal{P}} e_N(\mathcal{H}_{s, \alpha, \gamma, \mathcal{P}}),$$

where the infimum is extended over all $N$-element point sets $\mathcal{P}$ in $[0, 1]^s$.

Consider the (QMC) information complexity,

$$N_{\text{min}}(\varepsilon, s) = \min\{N \in \mathbb{N} : e(N, s) \leq \varepsilon\}.$$
We say that integration in $\mathcal{H}_{s,\alpha,\gamma}$ is

- weakly QMC tractable, if

$$\lim_{s+\varepsilon^{-1} \to \infty} \frac{\log N_{\min}(\varepsilon, s)}{s + \varepsilon^{-1}} = 0;$$

- polynomially QMC-tractable, if there exist $c, p, q \geq 0$ such that

$$N_{\min}(\varepsilon, s) \leq cs^q\varepsilon^{-p}. \quad (1)$$

Infima over all $q$ and $p$ such that (1) holds: $s$- and $\varepsilon$-exponent of polynomial tractability, respectively;

- strongly polynomially QMC-tractable, if (1) holds with $q = 0$. Infimum over all $p$ such that (1) holds: $\varepsilon$-exponent of strong polynomial tractability.
For the Korobov space $\mathcal{H}_{s,\alpha,\gamma}$ it is known that:

- $\sum_{j=1}^{\infty} \gamma_j < \infty$ is equivalent to strong polynomial tractability.
- If $\sum_{j=1}^{\infty} \gamma_j^{1/\tau} < \infty$ for some $\tau \in [1, \alpha)$, then one can set the $\varepsilon$-exponent to $2/\tau$.
- The $\varepsilon$-exponent of $2/\alpha$ is optimal.
- Use CBC-constructed lattice point sets to obtain optimal results.
Suppose now that

$$\sum_{j=1}^{\infty} \gamma_j^{1/\tau} < \infty$$

for some $\tau > \alpha$.

**So far:** CBC construction of lattice point sets that yield optimal $\varepsilon$-exponent, but cost of CBC-construction is independent of the weights.

**Our new result:** CBC construction of lattice point sets that yield optimal $\varepsilon$-exponent, but cost of CBC-construction may decrease with the weights.

Exploit situations where weights decrease sufficiently fast.
The reduced CBC construction
\textbf{Idea:} make search space smaller for later components.

- Let $N$ be a prime power, $N = b^m$, $b$ prime, $m \in \mathbb{N}$
- Let $w_1, \ldots, w_s \in \mathbb{N}_0$ with $0 = w_1 \leq \ldots \leq w_s$
- Consider the sequence of reduced search spaces

\[ Z_{N, w_j} := \begin{cases} 
\{ 1 \leq z < b^{m-w_j} : \gcd(z, N) = 1 \} & \text{if } w_j < m \\
\{ 1 \} & \text{if } w_j \geq m
\end{cases} \]

- Note that

\[ |Z_{N, w_j}| := \begin{cases} 
 b^{m-w_j-1}(b-1) & \text{if } w_j < m \\
1 & \text{if } w_j \geq m
\end{cases} \]

- write $Y_j := b^{w_j}$
The reduced CBC construction

Algorithm (Reduced CBC construction)

Let $N$, $w_1, \ldots, w_s$, and $Y_1, \ldots, Y_s$ be as above. Construct $z = (Y_1z_1, \ldots, Y_sz_s)$ as follows.

- Set $z_1 = 1$.
- For $d \leq s$ assume that $z_1, \ldots, z_{d-1}$ have already been found. Now choose $z_d \in \mathbb{Z}_{N,w_d}$ such that
  \[
e_{N,d,\alpha,\gamma}((Y_1z_1, \ldots, Y_dz_d, Y_dz_d))
\]
  is minimized as a function of $z_d$.
- Increase $d$ and repeat the second step until $(Y_1z_1, \ldots, Y_sz_s)$ is found.

Usual CBC construction: $w_j = 0$ and $Y_j = 1$ for all $j$. 

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Theorem

Let \( z = (Y_1 z_1, \ldots, Y_s z_s) \in \mathbb{Z}^s \) be constructed according to the reduced CBC algorithm. Then for every \( d \leq s \) it is true that

\[
e_{N, d, \alpha, \gamma}((Y_1 z_1, \ldots, Y_d z_d)) \leq \left( 2 \prod_{j=1}^{d} \left( 1 + \gamma_j \frac{1}{\alpha - 2\delta} 2\zeta \left( \frac{\alpha}{\alpha - 2\delta} \right) b^{w_j} \right) \right)^{\alpha/2-\delta} N^{-\alpha/2+\delta}
\]

for all \( \delta \in \left( 0, \frac{\alpha - 1}{2} \right] \), where \( \zeta \) is the Riemann zeta function.
Let $\delta \in (0, \frac{\alpha-1}{2}]$ and let $z$ be constructed according to the reduced CBC algorithm.

- If
  
  $$\lim_{s \to \infty} \frac{1}{s} \sum_{j=1}^{s} \gamma_j b^{w_j} = 0,$$

  then we have weak tractability.

- If
  
  $$A := \limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j^{\frac{1}{\alpha-2\delta}} b^{w_j}}{\log s} < \infty,$$

  then we have polynomial tractability with $\varepsilon$-exponent at most $\frac{2}{\alpha-2\delta}$ and $s$-exponent at most $2\zeta\left(\frac{\alpha}{\alpha-2\delta}\right)A$. 
• If

$$B := \sum_{j=1}^{\infty} \gamma_j^{\frac{1}{\alpha-2\delta}} b^w_j < \infty,$$

then we have strong polynomial tractability with $\varepsilon$-exponent at most $\frac{2}{\alpha-2\delta}$. 
The reduced fast CBC construction
The fast CBC construction (Nuyens/Cools) for the non-reduced case \( (w_j = 0) \) has a computation cost of \( \mathcal{O}(sN \log N) \).

The idea also works for the reduced case and yields reduced cost by exploiting additional structure of the case \( w_j > 0 \).

Bonus: once \( w_j \geq m \) the search space contains only one element. Thus the construction of additional components incurs no extra cost.

The computational cost of the reduced fast CBC construction is

\[
\mathcal{O} \left( N \log N + \min\{s, s^*\} N + \sum_{j=1}^{\min\{s, s^*\}} (m - w_j)Nb^{-w_j} \right),
\]

where \( s^* := \min\{j \in \mathbb{N} : w_j \geq m\} \).
Example:

- Suppose weights $\gamma_j$ are $\gamma_j = j^{-3}$.
- Fast CBC construction needs $O(smb^m)$ operations to compute a generating vector for which the worst-case error is bounded independently of the dimension.
- Reduced fast CBC construction: choose, e.g., $w_j = \lfloor \frac{3}{2} \log_b j \rfloor$.
- We need $O(mb^m + \min\{s, s^*\} mb^m)$ operations to compute a generating vector for which the worst-case error is still bounded independently of the dimension, as
  \[ \sum_j \gamma_j b^{w_j} < \zeta(3/2) < \infty. \]
- Reduced fast CBC construction significantly reduces computation cost.
The reduced fast CBC construction

Computation times and $\log_{10}$ worst case error for $b = 2$, $\alpha = 2$, $\gamma_j = j^{-3}$:

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<th>$s = 20$</th>
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$w_j = 0$ \hspace{1cm} $w_j = \left\lfloor \frac{3}{2} \log_b j \right\rfloor$
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Concluding remarks
• Reduced CBC constructions also works for general weights.
• Fast reduced CBC construction so far only for product weights.
• Everything (including fast construction for product weights) can be done analogously for a Walsh space with polynomial lattice points instead of lattice points.
• Instead of setting $z_j = 1$ if $w_j \geq m$, we can choose these $z_j$ at random. Error bound essentially stays the same.
• If $w_j \geq m$, we can even replace the components of the lattice point set by uniformly distributed random points. We then have a hybrid point set in the sense of Spanier, the error bound stays the same.
• Error in Korobov space can be related to error of suitably transformed lattice points in Sobolev spaces.
Concluding remarks

Thank you very much for your attention.